Phase Transitions in Stochastic Networks

Tong Zhu
Department of Statistics, The University of Auckland
t.zhu@auckland.ac.nz

Introduction

We are interested in analyzing the performance of a symmetric tree structured loss network model supporting both multilink (multicast) connections that connect a central node to its nearest neighbours and unilink (unicast) connections that connect just two adjacent neighbours (see Figure 1).

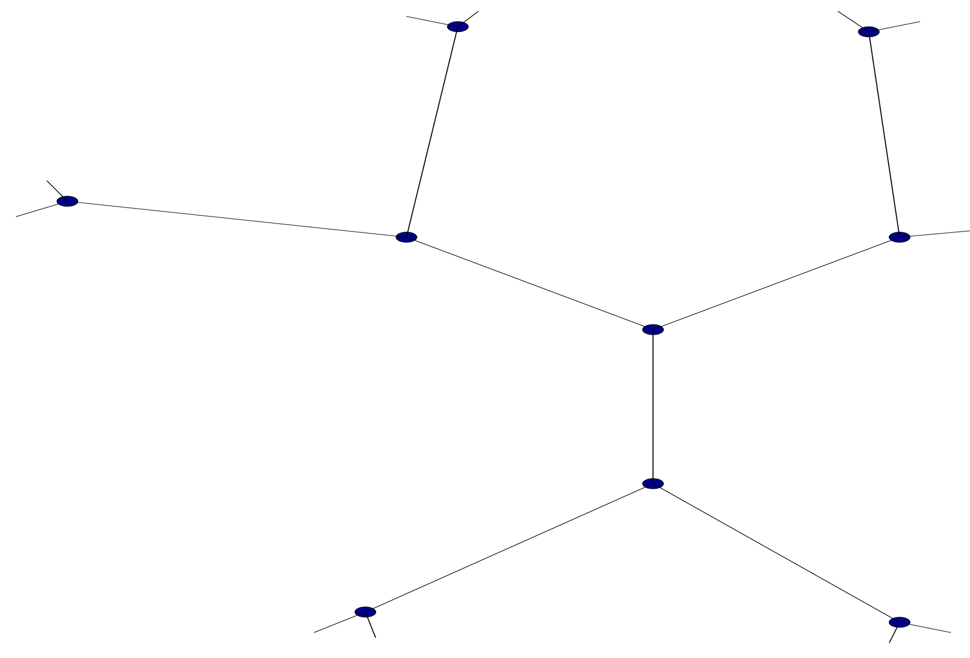

Figure 1: Loss network in the form of a spherical Cayley tree

Calls arrive in the network as a Poisson process and will be rejected or blocked if they cannot be served immediately due to insufficient capacities in the required links. Network performance is assessed by the probability that a call is blocked or lost (blocking probability). Such a network is considered as fair if the blocking probability for a certain type of call is uniform across the network space, however, phase transition effects can lead to unfairness or poor performance of the system. The critical questions are: When do phase transitions occur? How might they be avoided? Earlier work showed that phase transitions can occur in such networks with a symmetric tree structure and, more recently, that those phase transitions can be nonmonotone in the arrival rate of multilink connections at a node [1, 2].

Analysis of Performance

We break the symmetry of the network by randomly allocating two different link capacities, C_1 and C_2, across the network and analyzed the case $C_1 = 1$ and $C_2 = 2$. There are N_1 links with capacity C_1 and N_2 links with capacity C_2 for each node in the network. The finite network has a truncated product form stationary distribution with an explicit expression. Finding the normalizing constant of the stationary distribution can be infeasible as the network grows larger. However, due to the symmetry of the tree topology, we are able to develop a recursion scheme to evaluate the exact normalizing constant. Let $Z_m^{(j)}(i)$ represent the number of weighted sum of all feasible configurations that have i multicast calls in node with radius m whose rooted link has capacity j. Then the normalizing constant can be evaluated by the following recursions,

$$Z_{m+1}^{(j)}(i) = \frac{\nu^i}{i!} \cdot \left(\sum_{k_1=0}^{C_1-i} Z_m^{(C_1)}(k_1) \cdot \sum_{l_1=0}^{C_1-i-k_1} \frac{\lambda^{l_1}}{l_1!} \right)^{M_1(j)} \cdot \left(\sum_{k_2=0}^{\min(C_1,C_2-i)} Z_m^{(C_2)}(k_2) \cdot \sum_{l_2=0}^{C_2-i-k_2} \frac{\lambda^{l_2}}{l_2!} \right)^{M_2(j)}$$

where

$$M_i(j) := \begin{cases} \max(0,\ N_i - 1) & \text{if } j = C_i \\ \max(0,\ N_i) & \text{otherwise} \end{cases}$$

There is a connection between the stationary distribution of the finite network and the Markov random field on an infinite network. Moreover, phase transitions in the infinite network correspond to multi-mode of the stationary distribution for the finite network. Existence of multiple Gibbs measures in the infinite network are equivalent to the existence of multiple fixed points of the above recursion equations. We then are able to reduce the problem to the analysis of phase transitions for Gibbs measures that correspond to the product form Markov specifications with respect to the network.

Results

For a pure multicasting model, the network is equivalent to a collection of independent loss systems when $N_1 = 1$ and a collection of one-dimensional systems when $N_1 = 2$ respectively, independent of N_2. In both cases, phase transitions do not occur. When N_1 is greater than 2, the network is equivalent to a collection of symmetric loss networks with $C = 1$, $q = N_1 - 1$ in [2]. In that case, as shown by Ramanan et al. [2], phase transitions can occur.

Figure 2 shows the bifurcation map of a pure multicasting model with $N_1 = 3$ and $N_2 = 1$. It shows that when the multicast arrival rate is smaller than some critical point, there is a globally stable multicast blocking probability. However, this unique fixed blocking probability will lose its global stability at the critical point and a new stable two-period bifurcation exist.That is, phase transitions will occur at that critical point.

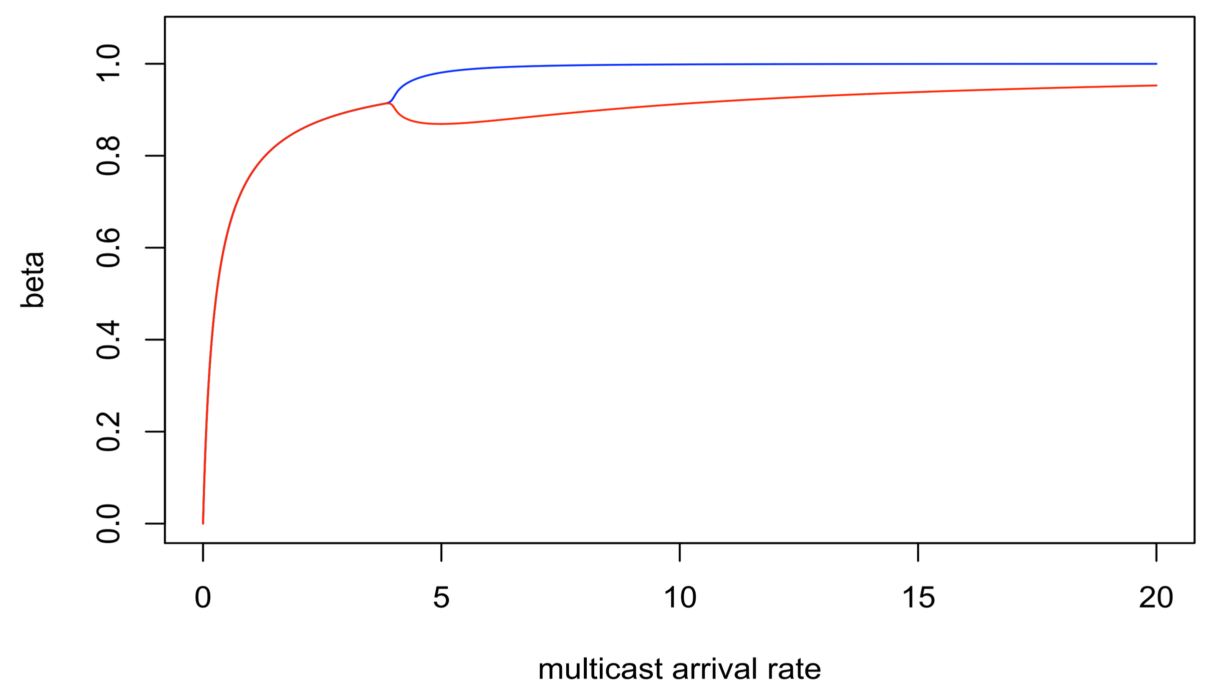

Figure 2: Multicast blocking probability for a loss network with $N_1 = 3$ and $N_2 = 1$

Acknowledgements

I would like to thank New Zealand Institute of Mathematics and its Applications (NZIMA) for its generosity in funding the Postgraduate student scholarship. I also would like to thank the University of Auckland for providing The University of Auckland International Tuition Fees Bursary scholarship. Thanks are also due to Dr. Ilze Ziedins, Dr. Vivien Kirk and Associate Professor David Scott for their guidance and time.

References

[1] Luen, B., Ramanan, K. and Ziedins, I. (2006). Nonmonotonicity of phase transitions in a loss network with controls. *Annals of Applied Probability* **3**, 1528-1562.

[2] Ramanan, K., Sengupta, A., Ziedins, I. and Mitra, P. (2002). Markov random field models of multicasting in tree networks. *Advances in Applied Probability* **34**, 58-84.