## Introduction

In the binary response model the probability, \( p_i \), of the success of an event, \( Y_i = 1 \), conditional on the covariate vector \( x_i \), is expressed as,

\[
y_i = \sum_{k=1}^n a_k \beta_k \quad i = 1, \ldots, n.
\]

The function \( g \) links the linear predictors to the probability and determines the shape of the quantal response. McCullagh and Nelder [3] represent four possible link functions for binary response model:

- **logit** : \( \log(\pi_i / (1 - \pi_i)) \)
- **probit** : \( \Phi^{-1}(\pi_i) \)
- **cauchit** : \( \text{cauchy}^{-1}(\pi_i) \)
- **cloglog** : \( \log(-\log(\pi_i)) \)

Koenker[2] implements two parametric families of link functions for binary response applications:

- **Gosset link** : the Student \( t \) link function to the corresponding link functions for the AIDS and AZT data sets.
- **Student link** (Pregibon[1]) : the Student \( t \) link function uses the distribution function of the Student \( t \) distribution.

The Student \( t \) link function is an alternative link function for the Poisson-ordinal link function (POLF), a negative binomial-ordinal link function (NBOLF), and a 2-parameter gamma (GOLF).

### The derivations

The link function is defined by

\[
y = g(\theta) = \beta^T x.
\]

The Student \( t \) link function uses the distribution function of the Student \( t \) random variables as the choice of the inverse of the link function. Therefore, \( \eta = F_{\nu}^{-1}(\pi) \) and \( \theta = F_{\nu}(\eta) \), where \( \theta \in (0,1) \) is the vector of parameters to be estimated. The first two derivatives of \( \theta \) with respect to \( \eta \) are

\[
\frac{\partial \eta}{\partial \theta} = F_{\nu}^{-1}(\eta), \quad \frac{\partial^2 \eta}{\partial \eta^2} = 0.
\]

The Student \( t \) link function

\[
\text{stlink}(\theta, \text{arg1} = \text{list}(\text{df} = 1), \text{inverse} = \text{FALSE}, \text{deriv} = 0, \text{short} = \text{TRUE}, \text{tag} = \text{FALSE})
\]

### The accuracy

- all.equal(stlink(p, probit(p)))
  [1] TRUE
- all.equal(stlink(p, cloglog(list(df=1)), cauchit(p))
  [1] TRUE
- all.equal(stlink(stlink(p), inverse = TRUE), p)
  [1] TRUE

### Future work

The next step is to investigate and develop a wider class of parametric link functions for binary and ordinal responses.

### Link functions for binary responses

The logit and probit are widely used in modeling for binary response data. However, they do not always provide the best fit available for a given data set. In the future frame work we will investigate and implement more flexible link functions such as

- **probit**
- **logit**
- **cloglog**

### Link functions for ordinal responses

Ye[6] develops three new link functions for ordinal responses obtained from the Poisson or negative binomial distribution: a Poisson-ordinal link function (POLF), a negative binomial-ordinal link function (NBOLF), and a 2-parameter gamma (GOLF).

The coefficients, log-likelihood, AIC, and the deviance obtained from glm() and vglm() with stlink() comparing to the corresponding link functions:

- all.equal(coef(Gosset), coef(stlink))
  [1] TRUE

### AIDS and AZT Data (Agresti[1])

The coefficients, log-likelihood, AIC, and the deviance obtained from glm() and vglm() with stlink() comparing to the corresponding link functions:

<table>
<thead>
<tr>
<th>Link function</th>
<th>Log-likelihood</th>
<th>AIC</th>
<th>Deviance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gosset(df = Inf)</td>
<td>-9.4383</td>
<td>24.8755</td>
<td>1.4003</td>
</tr>
<tr>
<td>probit()</td>
<td>-9.4383</td>
<td>24.8755</td>
<td>1.4003</td>
</tr>
<tr>
<td>stlink(df = Inf)</td>
<td>-9.4383</td>
<td>24.8755</td>
<td>1.4003</td>
</tr>
<tr>
<td>Gosset(df = 1)</td>
<td>-9.3339</td>
<td>24.6677</td>
<td>1.3914</td>
</tr>
<tr>
<td>cauchit()</td>
<td>-9.3339</td>
<td>24.6677</td>
<td>1.3914</td>
</tr>
</tbody>
</table>

### References


