THE USER OPTIMAL POLICY FOR A SIMPLE MARKOV TRANSPORTATION NETWORK.

Created by Heti Afimeimounga

Supervisor:
Dr. Ilze Ziedins
Statistic Dept., University of Auckland.

Advisor:-
Dr. Wiremu Solomon
Statistic Dept., University of Auckland.

1 Introduction

Delays in network where users individually choose the shortest route for themselves (user optimal policy) may be considerably greater than in the systems where a central controller directs users to the optimal routes. The reduction in performance is known as the cost of selfish routing.

The network model

- A much simplified network model of the choice that commuters face between a private (road, Q_R) and public (train, Q_T) transportation from a source to a destination is considered here (see Figure 1). Arriving general users have to choose between Q_R and Q_T so as to minimize their own expected delay.
- We consider when arriving general users made their choice given that they know the current state of the system (state-dependent routing) and when they do not know the current state of the system and made their choices probabilistically (probabilistic routing).

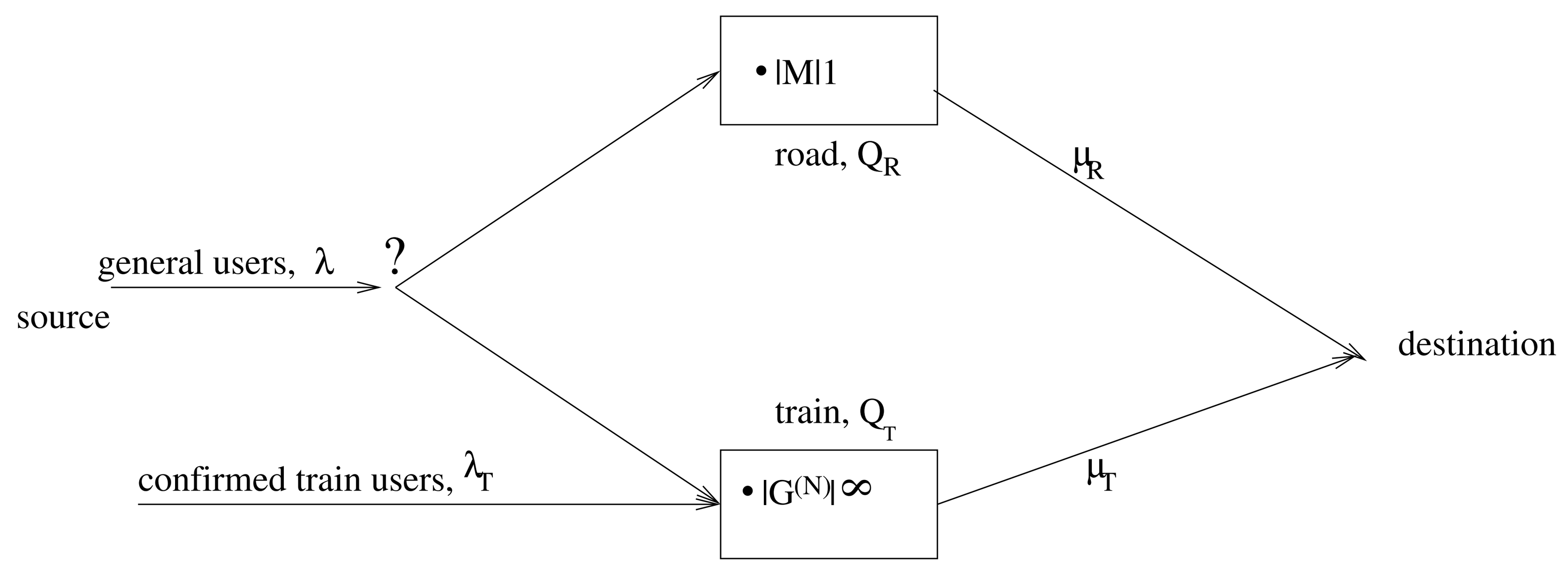

Figure 1: The network model

State-dependent routing

Let $y(\boldsymbol{x})$ and $z_D(\boldsymbol{x})$ denote the expected transit time for an arriving general commuter who sees $\boldsymbol{x} = (a, n) \in R$ and joins Q_R or sees $\boldsymbol{x} \in T$ and joins Q_T.

- The expected time in the system for a general user is given by

$$W = \sum_{\boldsymbol{x} \in \Omega} \pi_{\boldsymbol{x}} \Big(y(\boldsymbol{x}) I_{\boldsymbol{x} \in R} + z_D(\boldsymbol{x}) I_{\boldsymbol{x} \in T} \Big)$$

- Using sample path and coupling arguments in comparison of $y(\boldsymbol{x})$ and $z_D(\boldsymbol{x})$ we show that

Theorem

1. *Consider a process X, with parameters $\lambda, \lambda_T, \mu_R, \mu_T$ and $N \geq 2$. Then there exists a unique user optimal policy, $D^\star \in \mathcal{D}$ for this system. Furthermore, $D^\star$ is monotone.*
2. *$D^\star$ is monotone in parameters if $\lambda_1 > \lambda_2$, $\lambda_{T_1} > \lambda_{T_2}$, $\mu_{T_1} > \mu_{T_2}$, $\mu_{R_1} < \mu_{R_2}$.*

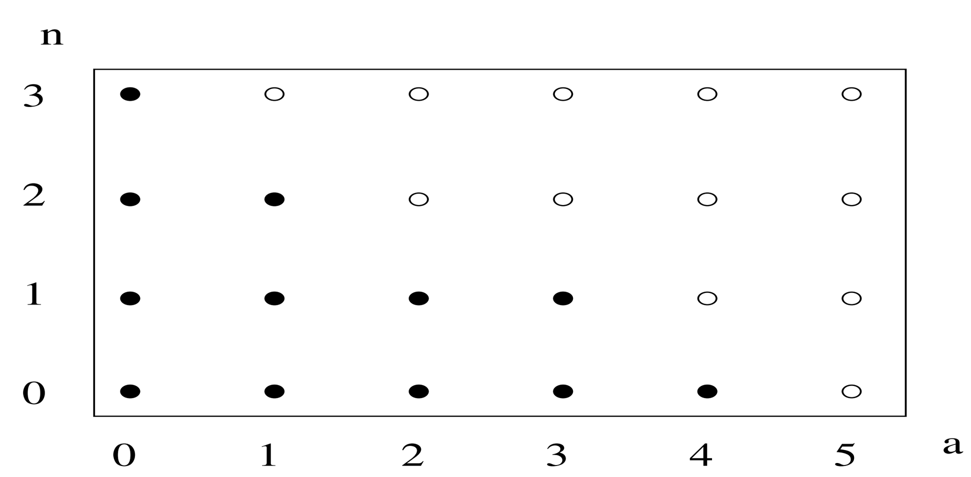

Figure 2: A monotone $D^\star = (R, T)$ when $N = 4$. $\boldsymbol{x} \in R$ are indicated by •. $\boldsymbol{x} \in T$ are indicated by ∘.

Comments: $D^\star$ is unique and monotone in all parameters, but W may not monotone in μ_R (see Figure 6) and λ.

Probabilistic Routing

- Arriving general user joins Q_R with probability p_R and Q_T with probability $1 - p_R$. p^{eq} is an user equilibrium if $W_T(p^{eq}) \leq W_R(p^{eq})$ or vice-versa, W_R = E(delay time via Q_R), W_T = E(delay time via Q_T). The expected transit time in the system for all general users is given by

$$W(p_R) = p_R W_R(p_R) + (1 - p_R) W_T(p_R)$$

Comment: Multiple user equilibria may exist for this routing case (see Figure 4).

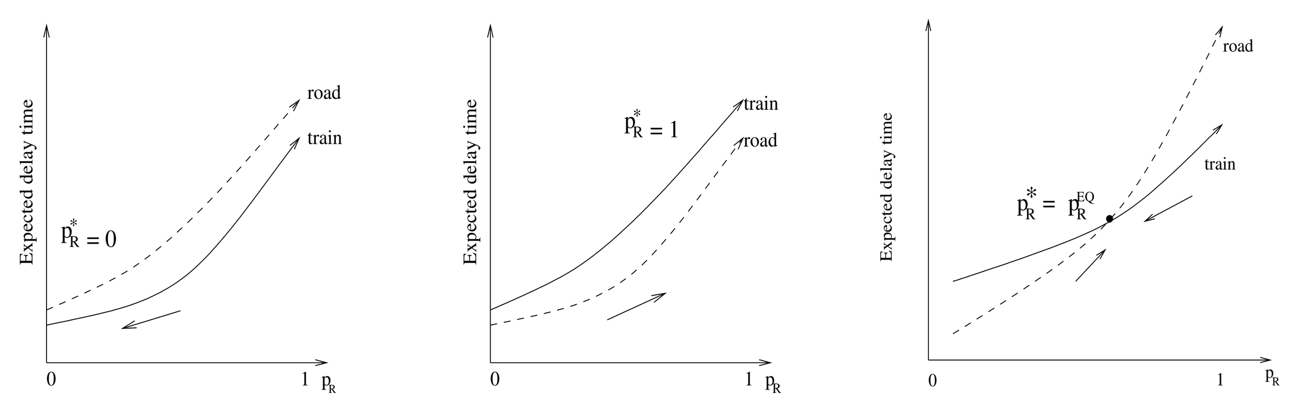

Figure 3: Unique stable user equilibria.

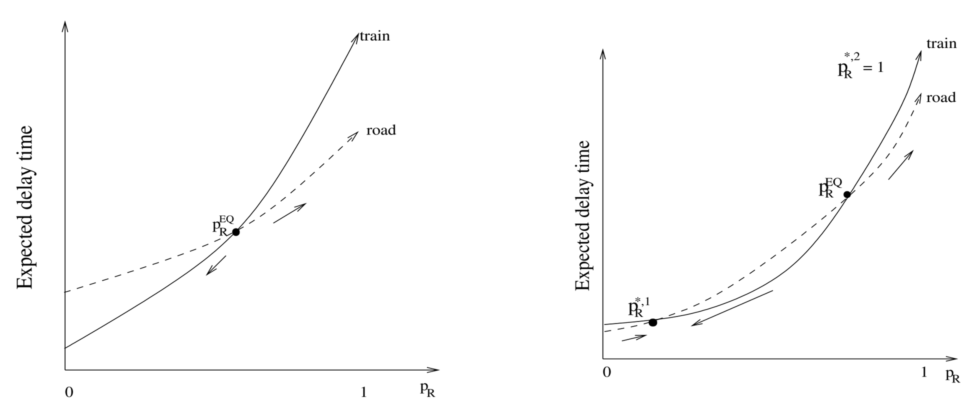

Figure 4: Multiple user equilibria.

2 Discussion

- W is not always decreasing as μ_R increases (Down-Thomson paradox) (see Figure 6) and it is observed under both state-dependent and probabilistic routing. State-dependent routing mitigates the effects of the Down-Thomson paradox observed under probabilistic routing, no sudden sharp increase and decrease when μ_R increases.

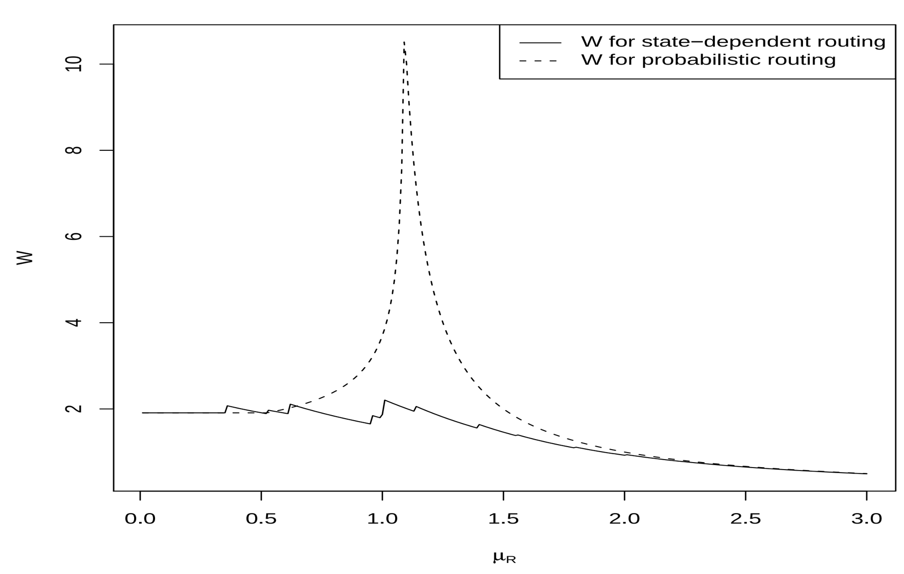

Figure 5: W vs. μ_R.

- If Q_T has one server, W_T is not monotone as in Q_T has infinite number of servers.

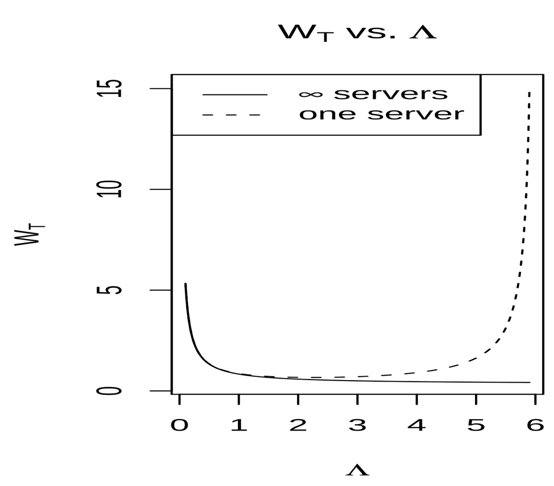

Figure 6: W vs. μ_R.

References

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