Auckland Mathematical Olympiad 2021

Junior Division

Questions

1. Solve the equation $\sqrt{x^2 - 4x + 13} + 1 = 2x$.

Solution. The equation is equivalent to

$$\sqrt{x^2 - 4x + 13} = 2x - 1.$$

and taking squares on both sides we get

$$x^2 - 4x + 13 = 4x^2 - 4x + 1$$

which is equivalent to $x^2 = 4$ or $x = \pm 2$.

When we take squares we may get roots that were not roots of the original equation since $x^2 = y^2$ is equivalent to $x = \pm y$ and not $x = y$. Thus we have to check our plausible roots substituting them in the original equation. Indeed -2 does not satisfy the original equation while 2 does.

Answer: the only solution is 2. □

2. Given five points inside an equilateral triangle of side length 2, show that there are two points whose distance from each other is at most 1.

Solution. Draw the equilateral triangle whose three vertices are the midpoints of the sides of the original triangle. This splits the original triangle into four equilateral triangles of side length 1. By the Pigeonhole Principle, one of these four triangles must contain two of the points, and the distance between those two points is at most 1. □

3. Alice and Bob are independently trying to figure out a secret password to Cathy's bitcoin wallet. Both of them have already figured out that:

- it is a 4-digit number whose first digit is 5;
- it is a multiple of 9;
- The larger number is more likely to be a password than a smaller number.

Moreover, Alice figured out the second and the third digits of the password and Bob figured out the third and the fourth digits. They told this information to each other but not actual digits. After that the conversation followed:

Alice: "I have no idea what the number is."

Bob: "I have no idea too."

After that both of them knew which number they should try first. Identify this number.

Solution. Let the password be $5XYZ$. As this number is a multiple of 9, we know that $5 + X + Y + Z$ is a multiple of 9. Alice knows $5 + X + Y$ and since she cannot identify the number it must be that $Z = 0$ or $Z = 9$. Similarly, $X = 0$ or $X = 9$. So the largest possibility is 5949. □

4. Four cars participate in a rally on a circular racecourse. They start simultaneously from the same point and go with a constant (but different) speeds. It is known that any three of them meet at some point. Prove that all four of them will meet again at some point.

Solution. Let cars be A, B, C, D and D is the slowest. Subtracting the speed of D from the speeds of all cars we may assume that D is in fact standing still on the starting point O. Since A, B, D meet, it means that A meets B at O after m laps. Similarly, A meets C at O after n laps. But then after mn laps A will meet at O both B and C. Since D is always there we have proved the statement. □

5. There are 13 stones each of which weighs an integer number of grams. It is known that any 12 of them can be put on two pans of a balance scale, six on each pan, so that they are in equilibrium (i.e., each pan will carry an equal total weight). Prove that either all stones weigh an even number of grams or all stones weigh an odd number of grams.

Solution. Let $x_1, \ldots, x_{13}$ be the stones and without loss of generality assume that x_{12}. and x_{13} have different parities. Let $X = x_1 + \ldots + x_{13}$. Leaving x_{13} aside we can split the remaining numbers into two equal sums: $x_{i_1} + \ldots + x_{i_6} = x_{j_1} + \ldots + x_{j_6}$. Hence $X - x_{13}$ is even. But similarly $X - x_{12}$ is even. This contradicts to x_{12}. and x_{13} having different parities. □

Auckland Mathematical Olympiad 2021

Senior Division

Questions

6. Find all real numbers x for which

$$\sqrt{\frac{x^3-8}{x}} > x-2.$$

Solution. Firstly, we have to figure out for which real x the left-hand-side makes sense. This happens in two cases:

$$\begin{cases} x^3-8 & \geq 0 \\ x & > 0 \end{cases} \quad \text{and} \quad \begin{cases} x^3-8 & \leq 0 \\ x & < 0 \end{cases}$$

Which happens when either $x \geq 2$ or $x < 0$. So solutions must be in $(-\infty, 0) \cup [2, \infty)$. We now work under this constraint.

We note that 2 is not a solution.

If $x-2 < 0$ or $x < 2$, the inequality is always satisfied since the right-hand-side is non-negative. Hence $(-\infty, 0)$ consists entirely of solutions.

If $x-2 > 0$ or $x > 2$, then both sides of inequality are positive and it will be equivalent to the inequality

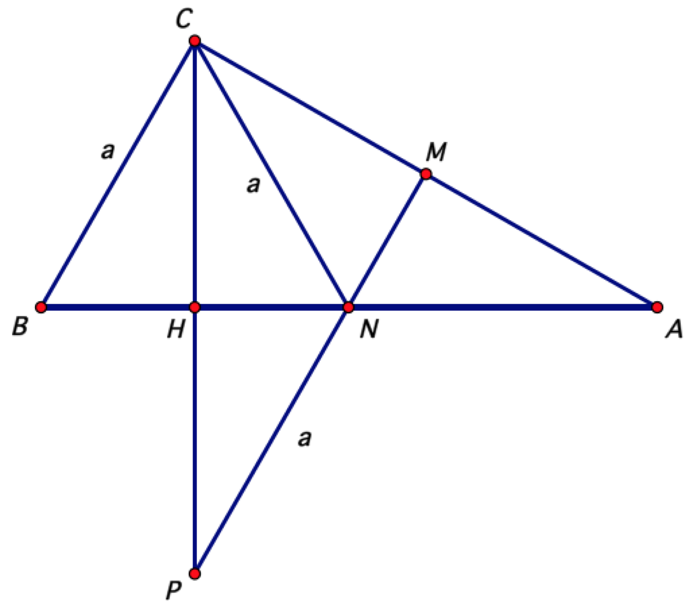
$$\frac{x^3-8}{x} > (x-2)^2.$$

which simplifies to $-4x^2+4x+8 < 0$ or $x^2-x-2 > 0$. The solution of the latter inequality is $(-\infty, -1) \cup (2, \infty)$. Given that we are under constraint $x > 2$, the solution in this case will be $(2, \infty)$.

Answer: $(-\infty, 0) \cup (2, \infty)$. □

7. Triangle ABC is the right angled triangle with the vertex C at the right angle. Let P be the point of reflection of C about AB. It is known that P and two midpoints of two sides of ABC lie on a line. Find the angles of the triangle.

Solution. Let C be the vertex at the right angle and CH be the altitude. The point P symmetric to C relative to AB is on line CH (continuation of the altitude) and $CH = PH$. Let M and N be the midpoints of AC and AB, respectively. Let us also draw the median CN. We have $CN = BN$ (indeed, N is the centre of the circumscribed circle about ABC).



Triangles $\triangle BCH$ and $\triangle NHP$ are equal, hence $PN = a$ and thus $CN = PN = a$. We also know that $BN = CN = a$, hence $\triangle BCN$ is equilateral.

Answer: $30°, 60°, 90°$. ☐

8. For how many integers n between 1 and 2021 does the infinite nested expression

$$\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{\cdots}}}}$$

give a rational number?

Solution. Denote the expression by x. Then $x^2 = n + x$ so $x^2 - x - n = 0$ and

$$x = \frac{1 \pm \sqrt{1+4n}}{2}.$$

This is rational if $1 + 4n = m^2$ or $4n = (m-1)(m+1)$. Hence m can be any odd number with $5 \le m^2 \le 8085$, i.e., $3 \le m \le 89$ (note $90^2 = 8100 > 8085$). There are 44 such values of m, corresponding to 44 values of n. ☐

9. Prove that there exist two powers of 7 whose difference is divisible by 2021.

Solution. There are 2021 remainders of division by 2021. Consider a sequence of powers $7^0 = 1, 7^1 = 7, 7^2, \ldots, 7^{2021}$. It contains 2022 members. Therefore, by the Pigeonhole principle, some two of them, say 7^n and 7^m, $n > m$, have equal remainders when divided by 2021. Then their difference $7^n - 7^m = 7^m(7^{n-m} - 1)$ is divisible by 2021. Since 7^m is relatively prime to 2021, then $7^{n-m} - 1 = 7^{n-m} - 7^0$ is divisible by 2021. ☐

10. There are 13 stones each of which weighs an integer number of grams. It is known that any 12 of them can be put on two pans of a balance scale, six on each pan, so that they are in equilibrium (i.e., each pan will carry an equal total weight). Prove that all stones weigh the same number of grams.

Solution. Let $x_1, \ldots, x_{13}$ be the non-negative integers representing the stones. If they satisfy the required condition, we will call them nice. We know that they are all even or they are all odd. Let us choose 13 nice non-negative integers, which are not all equal, with the smallest $\max_{i=1}^{13} x_i$. Note that, if we subtract the same number from $x_1, \ldots, x_{13}$ (by keeping results non-negative), then we again get a set of nice numbers. Therefore we may consider that $x_1 = 0$. But then all other numbers are even and divisible by 2. If we divide them by 2 we get again a set of nice non-negative integers which are not all equal. But this will contradict to the minimality of $\max_{i=1}^{13} x_i$. □