

# 2016 AUCKLAND MATHEMATICAL OLYMPIAD

## Questions

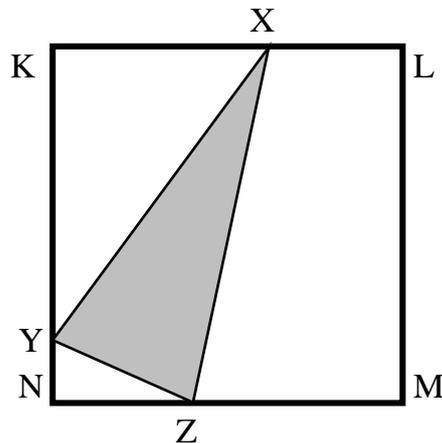
Write all your working and solutions below the question. You are expected to show how you obtained your solutions for each question.

### Junior Division

1. It is known that in a set of five coins three are genuine (and have the same weight) while two coins are fakes, each of which has a different weight from a genuine coin. What is the smallest number of weighings on a scale with two cups that is needed to locate one genuine coin?
2. The number 328 is written on the board. Two players alternate writing positive divisors of 328 on the board, subject to the following rules:
  - No divisor of a previously written number may be written;
  - The player who writes 328 loses.

Who has a winning strategy, the first player or the second player?

3. Triangle XYZ is inside square KLMN shown below so that its vertices each lie on three different sides of the square. It is known that:
  - The area of square KLMN is 1.
  - The vertices of the triangle divide three sides of the square up into these ratios:
$$\begin{aligned} KX : XL &= 3 : 2 \\ KY : YN &= 4 : 1 \\ NZ : ZM &= 2 : 3 \end{aligned}$$



What is the area of the triangle XYZ? (Note that the sketch is not drawn to scale).

4. If  $m$ ,  $n$ , and  $p$  are three different natural numbers, each between 2 and 9, what then are all the possible integer value(s) of the expression,  $\frac{m+n+p}{m+n}$  ?
5. A regular 2017-gon is partitioned into triangles by a set of non-intersecting diagonals. Prove that among those triangles only one is acute-angled.

## Senior Division

6. How many  $3 \times 5$  rectangular pieces of cardboard can be cut from a  $17 \times 22$  rectangular piece of cardboard, when the amount of waste is minimised?
7. In square  $ABCD$ ,  $\overline{AC}$  and  $\overline{BD}$  meet at point  $E$ .  
Point  $F$  is on  $\overline{CD}$  and  $\angle CAF = \angle FAD$ .  
If  $\overline{AF}$  meets  $\overline{ED}$  at point  $G$ , and if  $\overline{EG} = 24\text{cm}$ , then find the length of  $\overline{CF}$ .
8. In two weeks three cows eat all the grass on two hectares of land, together with all the grass that regrows there during the two weeks. In four weeks, two cows eat all the grass on two hectares of land, together with all the grass that regrows there during the four weeks.  
How many cows will eat all the grass on six hectares of land in six weeks, together with all the grass that regrows there over the six weeks?  
(Assume: - the quantity of grass on each hectare is the same when the cows begin to graze,  
- the rate of growth of the grass is uniform during the time of grazing,  
- and the cows eat the same amount of grass each week.)
9. Find the smallest positive value of  $36^k - 5^m$ , where  $k$  and  $m$  are positive integers.
10. In a city at every square exactly three roads meet, one is called street, one is an avenue, and one is a crescent. Most roads connect squares but three roads go outside of the city. Prove that among the roads going out of the city one is a street, one is an avenue and one is a crescent.

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## Solutions

1. It is easy to see that no single weighing gives the result. However 2 weighings would be enough. We compare weights of coins 1 and 2 and coins 3 and 4. If weights are equal both times, then coin 5 is genuine. If both unequal, then coin 5 is genuine again. If one is equal and another is not, then both coins in the equal weighing are genuine.

2. The first player has a winning strategy.

We have  $328 = 2^3 * 41$ .

Apart from divisors 1 and 328 we can split all other divisors into pairs:

$(2, 41), (2^2, 2*41), (2^3, 2^2 * 41)$ .

The first player writes 1 in his first move and then follows the following rule: if the second player chooses a number from one of the pairs, he replies with the second number of the same pair.

The first player always has a response, hence it is the second player who will eventually write 328 and lose.

3. The length of each side is 1.

$$\begin{aligned} \text{Area of grey triangle} &= \text{Area of square} \\ &\quad - (\text{Area of two white triangles} + \text{area of trapezium}) \\ &= 1 - \left[ \frac{1}{2} * \frac{3}{5} * \frac{2}{5} + \frac{1}{2} * \frac{4}{5} * \frac{1}{5} + \frac{1}{2} * \left( \frac{3}{5} + \frac{2}{5} \right) * 1 \right] \\ &= 1 - \left[ \frac{3}{25} + \frac{2}{25} + \frac{1}{2} \right] \\ &= \frac{3}{10} \text{ units}^2 \end{aligned}$$

4.  $\frac{m+n+p}{m+n} = 1 + \frac{p}{m+n}$ , but  $p$  must be a multiple of  $m+n$  for the entire fraction to be an integer.

The only possibility with  $m, n, p$  between 2 and 9 is that  $p = n+m$ , as if  $p = 2(n+m)$ , so  $p$  must be at least 10 and that contradicts the given values.

So, for all those values the fraction  $\frac{p}{m+n} = 1$

Hence the only entire value of  $\frac{m+n+p}{m+n} = 2$ .

5. The centre of the circumscribed circle is not on any of those diagonals (this must be checked). So it falls into one of the triangles. Only this triangle will have all its angles acute, all the other angles will have one obtuse angle (these must be also be demonstrated). If a solution omits obvious technical details, it should be given maybe 1 or 2 marks so their solution is expected to have more substance.

6. 24 is the maximum number of 3 x 5 cards that can be cut, since the amount of waste is then 14 square units, less than the area of one 3x5 card.  
There do exist configurations with 24 pieces.

7. Draw  $EK \parallel DC$  with  $K$  on  $AF$ . In right triangle  $AEG$ ,  $\angle AGE = 67.5$ ; and  $\angle EKG = \angle KAE + \angle AEK = 67.5$ .  
Hence triangle  $KEG$  is isosceles and  $FC = 2KE = 2EG = 48$

8. Let  $a$  be the original amount of grass on one acre, and let  $g$  stand for the amount of grass that grows on one acre in one week. Assume a cow eats  $e$  acres per week. Then we can say that the total amount of grass eaten under the first of the two given conditions is: number of cows \* amount eaten per week \* number of weeks

$$\begin{aligned} 3 * e * 2 &= 2a + 4g \\ 6e &= 2a + 4g \end{aligned}$$

And the second condition gives:  $2 * e * 4 = 2a + 4 * 2g$   
 $8e = 2a + 8g$

Subtracting the two obtained equations yields  $2e = 4g$ , so  $e = 2g$ .  
Plugging back into first equation gives  $12g = 2a + 4g$ , so  $a = 4g$ .

Now, the 6-week conditions asks how many cows will eat all the grass on six hectares of land in six weeks, together with all the grass that regrows there over the six weeks?

Let  $c$  represent the number of cows:

$$\begin{aligned} c * e * 6 &= 6a + 6 * 6g \\ 6ec &= 6a + 36g \end{aligned}$$

Plug in  $e = 2g$  and  $a = 4g$ , so  $6 * 2g * c = 6 * 4g + 36g$   
 $12cg = 24g + 36g = 60g$

So  $12c = 60$  and  $c = 5$ .

Answer: 5 cows

9.  $36^k = 6^{2k}$  always ends in 6, while  $5^m$  always ends in 5.  
So the difference  $36^k - 5^m$  always ends in 1. But it cannot be equal to 1.  
Indeed,  $6^{2k} - 1 - 5^m = (6^k - 1)(6^k + 1) - 5^m \neq 0$  since  $5^m$  is only divisible by 5, but 5 does not divide  $6^k + 1$ .  
Smallest value is 11, when  $k=1$  and  $m=2$ .

10. Let  $n$  be the number of squares. Let us split each road in two halves in the middle. Then the total number of halves of each type is even.  
Let  $x_1$ ,  $x_2$  and  $x_3$  be the number of streets, avenues and crescents leaving the city. Then we have  $n+x_1$ ,  $n+x_2$ ,  $n+x_3$  halves of each kind. These numbers are even, hence  $x_1$ ,  $x_2$ ,  $x_3$  have equal parity.  
But  $x_1+x_2+x_3 = 3$ , thus they are all odd, and  $x_1 = x_2 = x_3 = 1$ .